Journal of Sound and Vibration 329 (2010) 3171–3181

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

On the definition of the natural frequency of oscillations in nonlinear large rotation problems

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ARTICLE INFO

Article history:
Received 5 July 2009
Received in revised form
21 January 2010
Accepted 11 February 2010
Handling Editor: M.P. Cartmell
Available online 7 March 2010

ABSTRACT

Computational multibody system algorithms allow for performing eigenvalue analysis at different time points during the simulation to study the system stability. The nonlinear equations of motion are linearized at these time points, and the resulting linear equations are used to determine the eigenvalues and eigenvectors of the system. In the case of linear systems, the system eigenvalues remain the same under a constant coordinate transformation; and zero eigenvalues are always associated with rigid body modes, while nonzero eigenvalues are associated with non-rigid body motion. These results, however, cannot in general be applied to nonlinear multibody systems as demonstrated in this paper. Different sets of large rotation parameters lead to different forms of the nonlinear and linearized equations of motion, making it necessary to have a correct interpretation of the obtained eigenvalue solution. As shown in this investigation, the frequencies associated with different sets of orientation parameters can differ significantly, and rigid body motion can be associated with non-zero oscillation frequencies, depending on the coordinates used. In order to demonstrate this fact, the multibody system motion equations associated with the system degrees of freedom are presented and linearized. The resulting linear equations are used to define an eigevalue problem using the state space representation in order to account for general damping that characterizes multibody system applications. In order to demonstrate the significant differences between the eigenvalue solutions associated with two different sets of orientation parameters, a simple rotating disk example is considered in this study. The equations of motion of this simple example are formulated using Euler angles, Euler parameters and Rodriguez parameters. The results presented in this study demonstrate that the frequencies obtained using computational multibody system algorithms should not in general be interpreted as the system natural frequencies, but as the frequencies of the oscillations of the coordinates used to describe the motion of the system.

1. Introduction

In the linear theory of vibration, the system stability is examined using the eigenvalues that remain constant with time since the mass, damping and stiffness matrices are assumed to be constant [1–3]. Negative real parts of the eigenvalues are associated with stable modes, positive real parts are associated with unstable modes, and zero real parts are associated with modes that exhibit sustained oscillations. One mode with an eigenvalue that has a positive real part is sufficient to

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doi:10.1016/j.jsvi.2010.02.015

render the system unstable. In the case of a general damping matrix, a state space representation is often used to solve for the system eigenvalues and eigenvectors. In the case of linear systems, one can show that a constant coordinate transformation does not lead to a change in the eigenvalues, and as a consequence, conclusions on the stability and nature of oscillations obtained using one set of coordinates apply also when other sets of coordinates are used. Furthermore, in the case of linear systems, zero eigenvalues are always associated with rigid body modes; and nonzero eigenvalues are associated with non-rigid body motion.

In the case of nonlinear systems where the mass, damping, and/or the stiffness matrices do not remain constant; the eigenvalues and eigenvectors become configuration dependent and vary with time. The stability of nonlinear systems is often examined by linearizing the governing dynamic equations of motion at different system configurations. The resulting linearized equations are used to formulate a linear problem that can be solved for the eigenvalues and eigenvectors. The eigenvalues can be used to examine the system stability at the configurations at which the nonlinear equations are linearized. Frequencies of oscillations as well as damping ratios can be extracted from the solution of the eigenvalue problem in a straight forward manner. Unlike linear systems, as will be demonstrated in this paper, the eigenvalue solution depends on the set of coordinates used. Furthermore, rigid body motion can lead to non-zero eigenvalues, depending on the set of coordinates used. For this reason, the definition of the natural frequencies and interpretation of the eigenvalue solution of nonlinear systems is not as simple as in the case of linear systems. This issue is of particular significance in the study of the highly nonlinear multibody system applications.

The dynamics of multibody systems is governed by a system of differential and algebraic equations (DAEs). The differential equations represent the equations of motion of the system, while the nonlinear algebraic equations represent the kinematic constraints imposed on the motion of the system. General multibody system algorithms implemented in general purpose computer programs are designed to satisfy the constraint equations at the position, velocity, and acceleration levels. In order to solve for the eigenvalues and eigenvectors of the multibody system at different simulation time points, the constraint forces can be eliminated by writing the system accelerations in terms of the independent accelerations using a velocity transformation [4–8]. By eliminating the constraint forces, one obtains a minimum set of equations of motion associated with the system degrees of freedom. These equations can be linearized at different configurations in order to obtain a system of linear equations that can be used to formulate the eigenvalue problem. In order to account for the general damping matrix that characterizes most multibody system applications, the eigenvalue problem is formulated in multibody system algorithms using the state space approach.

In the multibody system applications, as previously mentioned, it is important to have a correct interpretation of the results of the eigenvalue solution. Different multibody system formulations employ different sets of orientation parameters. Some formulations employ Euler angles to describe the orientation of the body reference in space. In order to avoid the singularity problems associated with the use of the three Euler angles, some other multibody system formulations employ the four Euler parameters to describe the orientation of the body reference. The four Euler parameters are related by one nonlinear kinematic constraint equation that must be adjoined to the system equations of motion as an algebraic constraint equation. This constraint equation must be satisfied at the position, velocity and acceleration levels.

The purpose of this investigation is to demonstrate the significant difference between the eigenvalue solutions obtained when different sets of rotation parameters are used to describe the orientation of the body reference in space. In particular, the most widely used Euler angles and Euler parameters are employed in this investigation. Euler parameters are bounded, and therefore, simple free rotations that represent rigid body modes do not lead to zero frequency modes as in the case of Euler angles that can increase linearly if the system is torque free. For this reason, many of the concepts and conclusions drawn from the analysis of linear systems cannot be generalized and used in the case of nonlinear multibody systems. This problem is particularly important when comparing the vibration and stability results obtained using two different multibody system codes that employ the same reference frames but use different sets of parameters to define the orientations of these frames. The two computer codes can yield the same dynamics results and define the correct state of the system. The eigenvalue solutions obtained using the two codes, on the other hand, may look significantly different despite the fact that both solutions are correct and are associated with a correct system configuration. This paper addresses this important issue and explains the source of the differences between two eigenvalue solutions obtained using two different sets of orientation parameters that describe the motion of the same frame of reference. The paper also shows that the eigenvalues associated with rigid body modes can depend on the system initial conditions when a set of orientation parameters is used. For this reason, one should be careful in interpreting these eigenvalues as the system natural frequencies.

2. Background

In the case of linear vibration, the equations of motion of a mechanical system can be written in the following form [1,2]:

$$\mathbf{M}_q\ddot{\mathbf{q}} + \mathbf{K}_q\mathbf{q} = \mathbf{Q}_q \tag{1}$$

In this equation, $\mathbf{q}$ is the vector of system coordinates; $\mathbf{M}_q$ and $\mathbf{K}_q$ are, respectively, the constant symmetric system mass and stiffness matrices associated with the coordinates $\mathbf{q}$; and $\mathbf{Q}_q$ is the vector of generalized forces associated with $\mathbf{q}$. The vector of generalized forces $\mathbf{Q}_q$ is assumed to be independent of the coordinates and velocities, and therefore, such a vector does not contribute to the system stiffness or damping forces. Assume that the n-dimensional vector of coordinates $\mathbf{q}$ is related to another set of n coordinates $\mathbf{p}$ using the following linear relationships:

$$\mathbf{q} = \mathbf{B}_{qp}\mathbf{p}, \quad \dot{\mathbf{q}} = \mathbf{B}_{qp}\dot{\mathbf{p}}, \quad \ddot{\mathbf{q}} = \mathbf{B}_{qp}\ddot{\mathbf{p}} \tag{2}$$

In this equation, it is assumed that $\mathbf{B}_{qp}$ is a constant, square, and nonsingular velocity transformation matrix. Substituting Eq. (2) in Eq. (1), and pre-multiplying by the transpose of the velocity transformation matrix $\mathbf{B}_{qp}$, one obtains

$$\mathbf{M}_p\ddot{\mathbf{p}} + \mathbf{K}_p\mathbf{p} = \mathbf{Q}_p \tag{3}$$

In this equation, $\mathbf{M}_p$ and $\mathbf{K}_p$ are, respectively, the system mass and stiffness matrices associated with the coordinates $\mathbf{p}$; and $\mathbf{Q}_p$ is the vector of generalized forces associated with $\mathbf{p}$; they are defined as follows:

$$\mathbf{M}_p = \mathbf{B}_{qp}^T\mathbf{M}_q\mathbf{B}_{qp}, \quad \mathbf{K}_p = \mathbf{B}_{qp}^T\mathbf{K}_q\mathbf{B}_{qp}, \quad \mathbf{Q}_p = \mathbf{B}_{qp}^T\mathbf{Q}_q \tag{4}$$

In the case of free vibrations, Eq. (1) leads to the following generalized eigenvalue problem:

$$(\mathbf{K}_q - \omega_q^2\mathbf{M}_q)\mathbf{Y}_q = \mathbf{0} \tag{5}$$

where ω_q^2 is the eigenvalue (square of the natural frequency), and $\mathbf{Y}_q$ is the associated eigenvector. The eigenvalue problem associated with the free vibration of Eq. (3) is

$$(\mathbf{K}_p - \omega_p^2\mathbf{M}_p)\mathbf{Y}_p = \mathbf{0} \tag{6}$$

One can verify using Eq. (2) that

$$\omega_q^2 = \omega_p^2, \quad \mathbf{Y}_q = \mathbf{B}_{qp}\mathbf{Y}_p \tag{7}$$

Eq. (7) demonstrates that the use of a linear coordinate transformation does not change the system eigenvalues, and the new eigenvectors are a linear combination of the original eigenvectors, as it is known in linear algebra. That is, the choice of coordinates does not affect the eigenvalues in the case of linear vibration problems. Furthermore, regardless of the set of coordinates used, a zero eigenvalue is always associated with a rigid body mode; while a non-zero eigenvalue is associated with a non-rigid body motion.

3. Nonlinear large rotation problems

Most general multibody system computer codes allow for using a systematic procedure to linearize the highly nonlinear constrained differential equations of motion about nominal configurations at different time points specified by the user of the code. Quite often the eigenvalue results are used to study the system stability. Multibody system algorithms are designed to solve differential and algebraic equations (DAEs). The differential equations represent the equations of motion, while the algebraic equations represent the joint and specified trajectory constraints imposed on the motion of the system. Most general purpose multibody computer programs employ the technique of Lagrange multipliers to define the constraint forces. One method that can be used to solve the eigenvalue problem of the constrained multibody system at the time requested by the user of the code is to use the embedding technique at this point in time to eliminate the algebraic constraint equations and the associated Lagrange multipliers. This allows for writing the equations of motion in terms of the system degrees of freedom or independent coordinates $\mathbf{q}_i$. In this case, the equations of motion of the system can be written as [4–8]

$$\mathbf{M}_i(\mathbf{q}_i)\ddot{\mathbf{q}}_i = \mathbf{Q}_i(\mathbf{q}_i, \dot{\mathbf{q}}_i, t) \tag{8}$$

In this equation, $\mathbf{M}_i$ and $\mathbf{Q}_i$ are, respectively, the nonlinear mass matrix and generalized force vector associated with the system degrees of freedom $\mathbf{q}_i$. In general, Eq. (8) is a highly nonlinear matrix equation with a dense mass matrix. The source of nonlinearity in the preceding equation is due to the finite reference rotation, gyroscopic, Coriolis and contact forces, nonlinear spring, damping, and actuator coefficients, etc. In order to solve the eigenvalue problem at a certain nominal configuration that corresponds to a certain time point, a linear form of the preceding equation is obtained [7]. Quite often, the following form of the linearized free vibration equations of motion is used:

$$\mathbf{M}_i\ddot{\mathbf{q}}_i + \mathbf{C}_i\dot{\mathbf{q}}_i + \mathbf{K}_i\mathbf{q}_i = \mathbf{0} \tag{9}$$

In this equation, $\mathbf{M}_i$ is assumed to be known at the given configuration, $\mathbf{C}_i$ and $\mathbf{K}_i$ are damping and stiffness matrices defined, respectively, as

$$\mathbf{C}_i = -\frac{\partial \mathbf{Q}_i}{\partial \dot{\mathbf{q}}_i}, \quad \mathbf{K}_i = -\frac{\partial \mathbf{Q}_i}{\partial \mathbf{q}_i} \tag{10}$$

In the case of general damping matrix, the state space formulation can be used to determine the eigenvalues and eigenvectors of the multibody system. In this more general case, some of the eigenvalues and eigenvectors can be complex conjugates. Since the coordinates are real, a procedure for defining real mode shapes can be used as described in the literature.

The general procedure described in this section is used in this investigation to obtain the numerical results presented in Section 7 for a simple rotating system. Euler parameters are used to describe the orientation of the rotating body, and the kinematic constraints imposed on the motion of the system are introduced using nonlinear algebraic equations that are satisfied at the position, velocity, and acceleration levels. At the time specified to solve the eigenvalue problem, the embedding technique is used to eliminate the constraint equations and obtain a minimum set of differential equations. These equations, as described in this section, are linearized and used to define the eigenvalue problem that is solved for the system eigenvalues and eigen vectors.

4. Large rotations

Different sets of rotation parameters can be used to define the orientation of a rigid frame of reference in space. Among these sets are the three Euler angles and the four Euler parameters. Euler parameters are often used to avoid the singularity problem associated with the use of three parameter representations. The four Euler parameters, however, are related by one algebraic equation that must be introduced to the dynamic formulation as a kinematic constraint equation. This equation must be satisfied at the position, velocity and acceleration levels, as previously mentioned. In the following developments, body axes represent the axes of a rigid frame of reference whose origin is rigidly attached to the moving body or object. In the case of rigid body dynamics, a centrodial body coordinate system is used.

4.1. Euler angles

There are different sequences of Euler angles that can be found in the literature. In this investigation, a sequence that is widely used in vehicle dynamics is employed as an example [9]. The sequence consists of a rotation ψ (yaw) about the body Z axis, followed by a rotation ϕ (roll) about the body X axis, followed by a rotation θ (pitch) about the body Y axis. This sequence leads to the following transformation matrix:

$$\mathbf{A}_a = \begin{bmatrix} \cos\psi\cos\theta - \sin\psi\sin\phi\sin\theta & -\sin\psi\cos\phi & \cos\psi\sin\theta + \sin\psi\sin\phi\cos\theta \\ \sin\psi\cos\theta + \cos\psi\sin\phi\sin\theta & \cos\psi\cos\phi & \sin\psi\sin\theta - \cos\psi\sin\phi\cos\theta \\ -\cos\phi^i\sin\theta & \sin\phi & \cos\phi\cos\theta \end{bmatrix} \tag{11}$$

The rotation about the Y axis is selected the third in the sequence used in this investigation because in many vehicle applications including railroad vehicles, the yaw and roll are small angles, while the pitch (rotation of the wheels) increases with time. By selecting the pitch rotation to be the third in the sequence, the Euler angles' singularities can be avoided when the yaw and roll are small.

The absolute angular velocity vectors defined in the global and the body coordinate systems can be expressed, respectively, in terms of the derivatives of Euler angles as

$$\boldsymbol{\omega} = \mathbf{G}_a\dot{\boldsymbol{\theta}}, \quad \overline{\boldsymbol{\omega}} = \overline{\mathbf{G}}_a\dot{\boldsymbol{\theta}} \tag{12}$$

In this equation, $\boldsymbol{\theta} = [\psi \;\; \phi \;\; \theta]^T$, and

$$\mathbf{G}_a = \begin{bmatrix} 0 & \cos\psi & -\sin\psi\cos\phi \\ 0 & \sin\psi & \cos\psi\cos\phi \\ 1 & 0 & \sin\phi \end{bmatrix}, \quad \overline{\mathbf{G}}_a = \begin{bmatrix} -\cos\phi\sin\theta & \cos\theta & 0 \\ \sin\phi & 0 & 1 \\ \cos\phi\cos\theta & \sin\theta & 0 \end{bmatrix} \tag{13}$$

The absolute angular acceleration vectors defined in the global and body coordinate systems are given, respectively, in the case of Euler angles as

$$\boldsymbol{\alpha} = \mathbf{G}_a\ddot{\boldsymbol{\theta}} + \dot{\mathbf{G}}_a\dot{\boldsymbol{\theta}}, \quad \overline{\boldsymbol{\alpha}} = \overline{\mathbf{G}}_a\ddot{\boldsymbol{\theta}} + \dot{\overline{\mathbf{G}}}_a\dot{\boldsymbol{\theta}} \tag{14}$$

Note that when Euler angles are used, the angular acceleration vectors contain terms which are quadratic in the time derivatives of the angles.

4.2. Euler parameters

The four Euler parameters are used in multibody system algorithms in order to avoid the singularities associated with the three parameter representations. The four Euler parameters are denoted in this investigation as β_0, β_1, β_2, and β_3. The four Euler parameters are related by the constraint equation [5,6,8]

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \tag{15}$$

The transformation matrix expressed in terms of these parameters is defined as

$$\mathbf{A}_p = \begin{bmatrix} 1-2\beta_2^2-2\beta_3^2 & 2(\beta_1\beta_2-\beta_0\beta_3) & 2(\beta_1\beta_3+\beta_0\beta_2) \\ 2(\beta_1\beta_2+\beta_0\beta_3) & 1-2\beta_1^2-2\beta_3^2 & 2(\beta_2\beta_3-\beta_0\beta_1) \\ 2(\beta_1\beta_3-\beta_0\beta_2) & 2(\beta_2\beta_3+\beta_0\beta_1) & 1-2\beta_1^2-2\beta_2^2 \end{bmatrix} \tag{16}$$

Note that the degree of nonlinearity of this transformation matrix is much less than that of the matrix $\mathbf{A}_a$ obtained previously using Euler angles. The transformation matrix $\mathbf{A}_p$ does not contain explicitly trigonometric functions which are of infinite order.

The absolute angular velocity vectors defined in the global and body coordinate systems can be expressed in the forms of Eq. (12) as

$$\boldsymbol{\omega} = \mathbf{G}_p\dot{\boldsymbol{\beta}}, \quad \overline{\boldsymbol{\omega}} = \overline{\mathbf{G}}_p\dot{\boldsymbol{\beta}} \tag{17}$$

In this equation, $\boldsymbol{\beta} = [\beta_0 \ \ \beta_1 \ \ \beta_2 \ \ \beta_3]^T$, and

$$\mathbf{G}_p = 2\begin{bmatrix} -\beta_1 & \beta_0 & -\beta_3 & \beta_2 \\ -\beta_2 & \beta_3 & \beta_0 & -\beta_1 \\ -\beta_3 & -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}, \quad \overline{\mathbf{G}}_p = 2\begin{bmatrix} -\beta_1 & \beta_0 & \beta_3 & -\beta_2 \\ -\beta_2 & -\beta_3 & \beta_0 & \beta_1 \\ -\beta_3 & \beta_2 & -\beta_1 & \beta_0 \end{bmatrix} \tag{18}$$

The angular acceleration vectors defined in the global and body coordinate systems can be written, respectively, in terms of the derivatives of Euler parameters as

$$\boldsymbol{\alpha} = \mathbf{G}_p\ddot{\boldsymbol{\beta}}, \quad \overline{\boldsymbol{\alpha}} = \overline{\mathbf{G}}_p\ddot{\boldsymbol{\beta}} \tag{19}$$

Note that these expressions of the angular accelerations do not contain terms that are quadratic in the velocities since $\dot{\mathbf{G}}_p\dot{\boldsymbol{\beta}} = \dot{\overline{\mathbf{G}}}_p\dot{\boldsymbol{\beta}} = \mathbf{0}$.

5. Large rotation rigid body modes

In the case of nonlinear large rotation problems, rigid body motion is not always associated with zero frequency. The natural frequency of oscillations depends on the set of orientation parameters used. For this reason, it is important to recognize that the natural frequencies should not be interpreted as the system natural frequencies, but as the natural frequencies of oscillations of the coordinates used to describe the motion of the system. Different coordinates lead to different forms of the equations of motion and to different forms of the linearized equations used to solve the eigenvalue problem. This fact is demonstrated in this section using a simple one degree of freedom example. The equation of motion of the system is formulated using Euler angles and Euler parameters. The three-dimensional Newton–Euler equations, which are expressed in terms of the angular velocity and acceleration vectors, can be used as the starting point in the formulation of the equation of motion of the simple system.

5.1. Newton–Euler equations

The Newton–Euler equations that govern the motion of a rigid body in space is given by [4–7]

$$\begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \overline{\mathbf{I}}_{\theta\theta} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{R}} \\ \overline{\boldsymbol{\alpha}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_e \\ \overline{\mathbf{M}}_e - \overline{\boldsymbol{\omega}} \times (\overline{\mathbf{I}}_{\theta\theta}\overline{\boldsymbol{\omega}}) \end{bmatrix} \tag{20}$$

where m is the mass of the rigid body, $\mathbf{R}$ is the vector that describes the translation of the body reference, $\mathbf{I}$ is a 3×3 identity matrix, $\overline{\mathbf{I}}_{\theta\theta}$ is the inertia tensor defined with respect to the centroidal body coordinate system, $\mathbf{F}_e$ is the resultant of the external forces, and $\overline{\mathbf{M}}_e$ is the resultant of the external moments defined in the body coordinate system. In the case of pure rotational motion, $\ddot{\mathbf{R}} = \mathbf{0}$. In this special case, the preceding equation reduces to Euler equation given by

$$\overline{\mathbf{I}}_{\theta\theta}\overline{\boldsymbol{\alpha}} = \overline{\mathbf{M}}_e - \overline{\boldsymbol{\omega}} \times (\overline{\mathbf{I}}_{\theta\theta}\overline{\boldsymbol{\omega}}) \tag{21}$$

This equation can be further simplified in the case of a rotation about a single axis. In the case of a rotation about a single axis, the gyroscopic forces are equal to zero. In the remainder of this section, the rotation about the body Y axis is considered. In this section, Euler angles and Euler parameters are considered. In the following section, the linearized equation is also derived using *Rodriguez parameters* in order to demonstrate the effect of the choice of the orientation parameters on the form of the linearized equations.

5.2. Linearized Euler angle equations

In the case of a single rotation about the body Y axis ($\psi = \phi = 0$), one can show that the angular velocity vector $\overline{\boldsymbol{\omega}}$ and angular acceleration vector $\overline{\boldsymbol{\alpha}}$ defined in the body coordinate system can be written using Eqs. (12) and (14), respectively, as

$$\overline{\boldsymbol{\omega}} = \mathbf{G}_{ar}\dot{\theta}, \quad \overline{\boldsymbol{\alpha}} = \mathbf{G}_{ar}\ddot{\theta} \tag{22}$$

where $\mathbf{G}_{ar}$ is a column vector defined as $\mathbf{G}_{ar} = [0 \ \ 1 \ \ 0]^T$. Substituting Eq. (22) in Eq. (21) and pre-multiplying by the transpose of the velocity transformation matrix $\mathbf{G}_{ar}$, one obtains the familiar single rotational equation that governs the planar motion of the single degree of freedom system as

$$I_{yy}\ddot{\theta} = \mathbf{G}_{ar}^T\overline{\mathbf{M}}_e \tag{23}$$

In this equation, I_{yy} is the mass moment of inertia about the body Y axis. If $\overline{\mathbf{M}}_e$ is assumed to be zero (torque free) or if it does not depend on the coordinates or velocities, then it will not contribute to the stiffness or damping when the equation of motion is linearized. Using this assumption, one can show that the linearized equation in the case of Euler angles is given as $I_{yy}\ddot{\theta}=0$. This equation has zero damping and stiffness coefficients and has one zero eigenvalue that corresponds to a rigid body mode associated with motion of the body about its axis of rotation. This eigenvalue solution is the expected results of a disk rotating freely about its axis of rotation. For given initial conditions $\theta(t=0)=\theta_0$ and $\dot{\theta}(t=0)=\dot{\theta}_0$, where t is time, θ in the case of a torque free motion increases linearly according to the equation $\theta=\theta_0+\dot{\theta}_0 t$. That is, there is no bound on the angle of rotation θ. This simple example shows that the rigid body mode is associated with a zero eigenvalue when Euler angles are used to formulate the dynamic equations of motion of the system.

5.3. Linearized Euler parameter equations

Recall that Euler parameters can be written in terms of the components of the axis of rotation $\mathbf{v}=[v_1\ \ v_2\ \ v_3]^T$, and the angle of rotation θ as [5,8]

$$\beta_0=\cos\frac{\theta}{2},\quad \beta_1=v_1\sin\frac{\theta}{2},\quad \beta_2=v_2\sin\frac{\theta}{2},\quad \beta_3=v_3\sin\frac{\theta}{2} \tag{24}$$

In the case of a simple rotation about the body Y axis, $\beta_1=\beta_3=0$. In this case, the Euler parameters constraint of Eq. (15) reduces to $\beta_0^2+\beta_2^2=1$. Using this constraint equation, one can write β_0 and its derivatives in terms of β_2 as

$$\left.\begin{aligned} \beta_0=\sqrt{1-\beta_2^2},\quad \dot{\beta}_0=-\frac{\beta_2\dot{\beta}_2}{\sqrt{1-\beta_2^2}},\\ \frac{\partial\beta_0}{\partial\beta_2}=-\frac{\beta_2}{\sqrt{1-\beta_2^2}},\quad \ddot{\beta}_0=-\frac{\beta_2\ddot{\beta}_2}{\sqrt{1-\beta_2^2}}+\gamma_\beta \end{aligned}\right\} \tag{25}$$

In this equation, $\gamma_\beta=-(\dot{\beta}_0^2+\dot{\beta}_2^2)/\beta_0$. Using the results of Eq. (25), one can write Euler parameters accelerations in terms of the independent Euler parameter acceleration $\ddot{\beta}_2$ as

$$\ddot{\boldsymbol{\beta}}=\mathbf{B}_{pr}\ddot{\beta}_2+\boldsymbol{\gamma}_\beta \tag{26}$$

In this equation, $\mathbf{B}_{pr}$ and $\boldsymbol{\gamma}_\beta$ are column vectors defined as

$$\mathbf{B}_{pr}=[-(\beta_2/\beta_0)\ \ 0\ \ 1\ \ 0)]^T,\quad \boldsymbol{\gamma}_\beta=[\gamma_\beta\ \ 0\ \ 0\ \ 0]^T \tag{27}$$

Using Eqs. (17), (19), (20), and (26); one can show that the equation of motion of the single degree of freedom system expressed in terms of Euler parameters is given by

$$\ddot{\beta}_2+(\dot{\beta}_0^2+\dot{\beta}_2^2)\beta_2=0 \tag{28}$$

The following comments apply to Eq. (28):

1. Since Euler parameters are bounded, the equation of motion of a freely rotating body must include positive stiffness and/or damping coefficients in order to ensure that the maximum values of Euler parameters do not exceed one. For this reason, the form of Eq. (28) is significantly different from the form of the same equation expressed in terms of Euler angles.
2. By using the definitions of Euler parameters given by Eq. (24), one can show that Eq. (28) reduces to the same equation $\ddot{\theta}=0$ obtained when Euler angles are used. Therefore, the dynamics defined by the two equations are in principle the same despite the fact that the two equations have significantly different forms.
3. If the angular velocity remains constant, one can show that the coefficient $(\dot{\beta}_0^2+\dot{\beta}_2^2)$ of β_2 in Eq. (28) remains constant. Using the definition of Euler parameters, it is clear that $\dot{\beta}_0^2+\dot{\beta}_2^2=(\dot{\theta}^2/4)(\sin^2\theta+\cos^2\theta)=\dot{\theta}^2/4$. If the angular velocity varies with time, the coefficient $(\dot{\beta}_0^2+\dot{\beta}_2^2)$ will not remain constant and will vary with time.
4. Euler parameter β_0 can be systematically eliminated from Eq. (28) by using the results of Eq. (25), leading to another form of the equation of motion expressed in terms of the degree of freedom β_2. The resulting equation $\ddot{\beta}_2+(1-\beta_2^2)^{-1}\dot{\beta}_2^2\beta_2=0$ is highly nonlinear.
5. While the initial conditions do not affect the rigid body mode when the equations are expressed in terms of Euler angles, Eq. (28) shows that the initial conditions can have a significant effect on the second term in this equation, and therefore, the resulting eigenvalue solution strongly depends on the initial conditions as will be demonstrated in this paper by a numerical example.

In order to obtain the linearized equation, let $Q_i = -(\dot{\beta}_0^2 + \dot{\beta}_2^2)\beta_2$. It follows that the stiffness and damping coefficients associated with the degree of freedom β_2 are given, respectively, by

$$\left.\begin{aligned} k_i &= -\frac{\partial Q_i}{\partial \beta_2} = (\dot{\beta}_0^2 + \dot{\beta}_2^2) + \beta_2 \frac{\partial \dot{\beta}_0^2}{\partial \beta_2} = \dot{\beta}_0^2 + \dot{\beta}_2^2 + \frac{2\dot{\beta}_0^2}{\beta_0^2} \\ c_i &= -\frac{\partial Q_i}{\partial \dot{\beta}_2} = 2\beta_2\dot{\beta}_2 + 2\beta_2\dot{\beta}_0 \frac{\partial \dot{\beta}_0}{\partial \dot{\beta}_2} = \frac{2\beta_2}{\beta_0}(\dot{\beta}_2\beta_0 - \dot{\beta}_0\beta_2) \end{aligned}\right\} \tag{29}$$

The linearized equation then takes the form

$$\ddot{\beta}_2 + c_i\dot{\beta}_2 + k_i\beta_2 = 0 \tag{30}$$

It is clear from the preceding two equations that the stiffness and damping coefficients are configuration dependent, and they also depend on the initial conditions. One can also write the stiffness and damping coefficients k_i and c_i in terms of the angle of rotation θ as

$$k_i = \frac{\dot{\theta}^2}{4}\left(1 + 2\tan^2\frac{\theta}{2}\right), \quad c_i = \dot{\theta}\tan\frac{\theta}{2} \tag{31}$$

Note that in this simple example, $\dot{\theta} = 2(\dot{\beta}_2\beta_0 - \dot{\beta}_0\beta_2)$.

Eqs. (29) and (31) show that singularities can be encountered when β_0 approaches zero. Multibody system algorithms are designed to allow for the change of the system degrees of freedom to avoid singularities and ill-conditioned matrices. In the computer implementation, the numerical properties of the constraint Jacobian matrix are used to determine the optimum set of independent coordinates or the system degrees of freedom. Therefore, when singularities are encountered in case β_2 is used as the system degree of freedom, the computer code automatically selects β_0 as the degree of freedom and evaluates the stiffness and damping coefficients associated with β_0.

6. Rodriguez parameters

Another example of a set of orientation coordinates that can lead to eigenvalue results different from the results obtained using Euler angles and Euler parameters is the set of Rodriguez parameters. The three Rodriguez parameters, which are not bounded, are defined in terms of the angle of rotation θ and the components of the unit vector $\mathbf{v} = [v_1 \ \ v_2 \ \ v_3]^T$ as [5,8]

$$\gamma_1 = v_1\tan\frac{\theta}{2}, \quad \gamma_2 = v_2\tan\frac{\theta}{2}, \quad \gamma_3 = v_3\tan\frac{\theta}{2} \tag{32}$$

It is clear from this definition that when $\theta=\pi$, singularities are encountered when Rodriguez parameters are used. In the disk example discussed in this paper, one has $\gamma_1 = 0,\ \gamma_2 = \tan(\theta/2),\ \gamma_3 = 0$. Using the equations of motion $\ddot{\theta} = 0$, and the Rodriguez parameter identity $\sec^2(\theta/2) = 1 + \gamma^2$, where $\gamma^2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \gamma_2^2$ in this example, one can show that the equation of motion of the disk can be written as

$$(1 + \gamma_2^2)\ddot{\gamma}_2 = 2\gamma_2\dot{\gamma}_2^2 \tag{33}$$

This equation leads to

$$\frac{d\dot{\gamma}_2}{\dot{\gamma}_2} = \frac{2\gamma_2}{1 + \gamma_2^2}d\gamma_2 \tag{34}$$

Integrating this equation, one obtains

$$\ln(\dot{\gamma}_2) = \ln(1 + \gamma_2^2) + \ln(c) = \ln(c(1 + \gamma_2^2)) \tag{35}$$

In this equation c is the constant of integration. It follows that

$$\dot{\gamma}_2 = c(1 + \gamma_2^2), \tag{36}$$

which leads to the following integral

$$\int \frac{d\gamma_2}{(1 + \gamma_2^2)} = c\int dt \tag{37}$$

Using integration tables, one has

$$\int \frac{d\gamma_2}{(1 + \gamma_2^2)} = \tan^{-1}\gamma_2 - c_1 \tag{38}$$

where c_1 is a constant. It follows from the preceding two equations that $\tan^{-1}\gamma_2 = ct + c_1$ or

$$\gamma_2 = \tan(ct + c_1) \tag{39}$$

It is clear from this equation that the Rodriguez parameter γ_2 exhibits in this special case of rigid body motion of the disk a behavior different from Euler angles and Euler parameters.

Using the linearization procedure described previously in this paper, one can show that the linearized equation of Eq. (33) can be written as

$$m_i\ddot{\gamma}_2 + c_i\dot{\gamma}_2 + k_i\gamma_2 = 0 \tag{40}$$

The coefficients in this equation are

$$m_i = 1 + \gamma_2^2, \quad c_i = -4\gamma_2\dot{\gamma}_2, \quad k_i = -2\dot{\gamma}_2^2 \tag{41}$$

Since the coefficient k_i in this equation remains negative, while m_i is positive, the eigenvalues of this system will always remain real. The solution, therefore, will be non-oscillatory leading to results which are different from the results obtained using Euler angles and Euler parameters. It is also clear from the preceding two equations that for a given set of initial conditions one obtains non-zero eigenvalue associated with rigid body motion of the disk if Rodriguez parameters are used.

7. Numerical results

The general procedure described in Section 3 is implemented in the general purpose multibody system computer code SAMS/2000 [7] which is used in this investigation to obtain the numerical results presented in this section for a simple rotating system. Euler parameters are used to describe the orientation of the rotating body, and the kinematic constraints imposed on the motion of the system are introduced using nonlinear algebraic equations that are satisfied at the position, velocity, and acceleration levels. These kinematic equations include the Euler parameter constraint equation defined by Eq. (15). At the time specified to solve the eigenvalue problem, the embedding technique is used to eliminate the constraint equations and obtain a minimum set of differential equations. These equations, as described in Section 3, are linearized and used to define the eigenvalue problem that is solved for the system eigenvalues and eigen vectors. In the case of a single degree of freedom system, the embedding technique leads to a single differential equation of motion.

A circular disk rotating freely about its axis experiences a rigid body motion. When Euler angles are used, this motion is governed by the equation $\ddot{\theta} = 0$, which shows that, for a given initial velocity and in the absence of torque, θ grows linearly. When Euler parameters are used, the motion of the same system is governed by the equation $\ddot{\beta}_2 + (\dot{\beta}_0^2 + \dot{\beta}_2^2)\beta_2 = 0$ (Eq. (28)), as previously discussed in this paper. Fig. 1 shows the solution of Eq. (28) when the disk is given an initial angular velocity of 100 rad/s. The results presented in this figure show that β_0 and β_2 remain bounded and are oscillatory despite the fact that the disk experiences a rigid body motion. In this study, β_0 is obtained from the degree of freedom β_2 by using the Euler parameter constraint. Fig. 2 shows $\dot{\beta}_0$ and $\dot{\beta}_2$ as function of time. The sum of these two derivatives defines the coefficient of β_2 in the equation of motion. Since the system is torque free, the angular velocity of the disk remains constant, and as previously mentioned in this paper, the coefficient $(\dot{\beta}_0^2 + \dot{\beta}_2^2)$ remains constant and is equal to $(\dot{\theta}^2/4) = 2500$ in this particular example.

Similar results in the case of an initial angular velocity of 200 rad/s are reported in Figs. 3 and 4. In particular, the results of Fig. 4 show that the amplitudes of the derivatives increase as compared to the amplitudes of the derivatives presented in

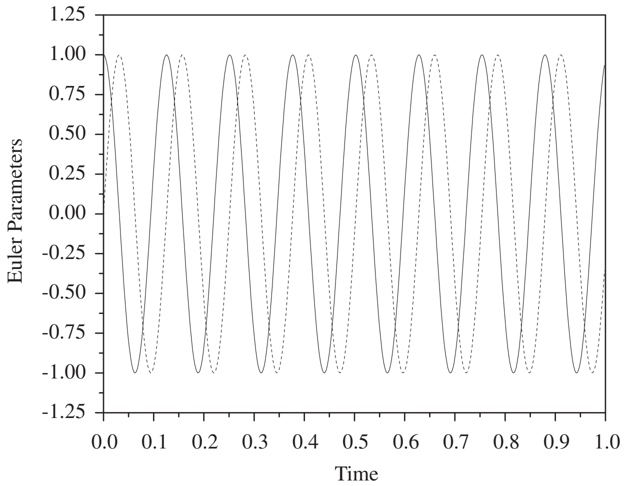

Fig. 1. Euler parameters for $\dot{\theta} = 100$ rad/s (——— β_0, - - - - β_2).

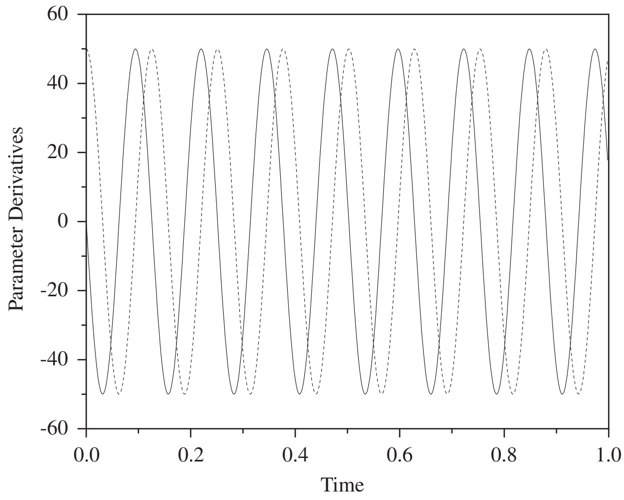

Fig. 2. Derivatives of Euler parameters for $\dot{\theta} = 100\,\text{rad/s}$ (——— $\dot{\beta}_0$, - - - - $\dot{\beta}_2$).

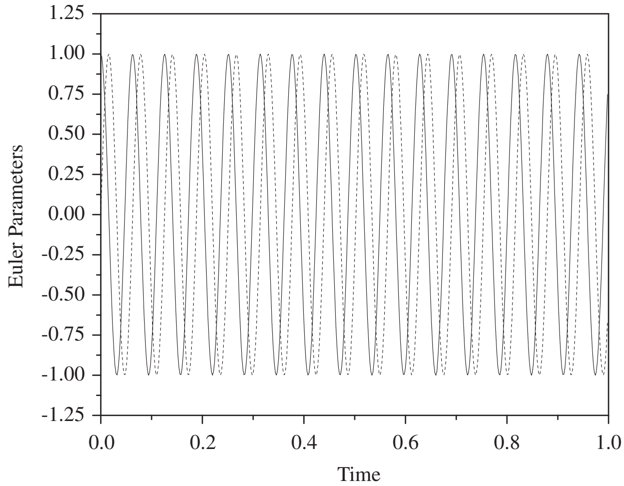

Fig. 3. Euler parameters for $\dot{\theta} = 200\,\text{rad/s}$ (——— β_0, - - - - - - β_2).

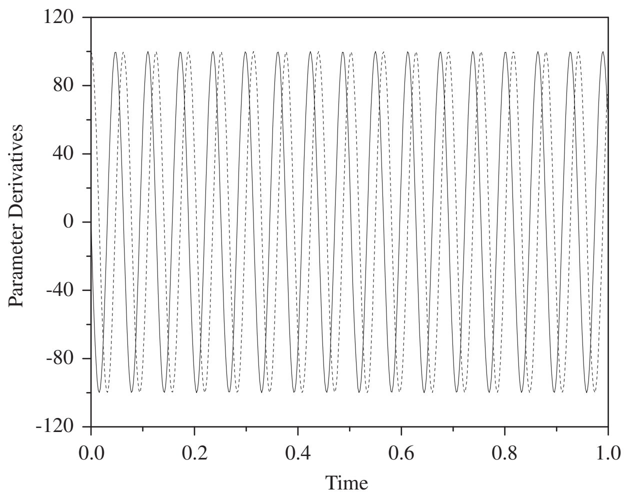

Fig. 4. Derivatives of Euler parameters for $\dot{\theta} = 200\,\text{rad/s}$ (——— $\dot{\beta}_0$, - - - - - - $\dot{\beta}_2$).

Table 1
Eigenvalues of the rotating disk.

Time	Angular velocity of 100 rad/s		Angular velocity of 200 rad/s	
	Real part	Imaginary part	Real part	Imaginary part
0.000	0.000	50.000	0.000	100.000
0.091	8.191	51.977	33.661	115.745
0.182	16.833	57.877	75.903	165.151
0.273	26.488	67.846	−67.970	154.471
0.364	37.963	82.653	−27.964	111.113
0.455	−47.422	96.084	5.164	100.403
0.546	−33.964	77.212	39.456	121.133
0.637	−23.199	64.167	84.170	176.780
0.728	−13.959	55.537	−60.912	145.381
0.819	−5.521	50.908	−22.657	107.423
0.910	2.617	50.206	10.172	101.546

Fig. 2. In this case of higher initial angular velocity, the stiffness coefficient $(\dot{\beta}_0^2+\dot{\beta}_2^2)$ in Eq. (28) becomes significantly larger and is equal to $(\dot{\theta}^2/4)=1000$. These results indicate that the eigenvalue obtained when using Euler parameters depend on the configuration as well as the initial conditions. Table 1 shows the real and imaginary parts of the eigenvalues determined by linearization of Eq. (28) for the two initial angular velocities considered in this section. The results presented in this table clearly show that the eigenvalue that corresponds to a rigid body motion of the disk is not zero when Euler parameters are used. Furthermore, the real part of the eigenvalue is not zero and varies with time since Euler parameters oscillate.

While a simple numerical example is used in this section to demonstrate the dependence of the natural frequency on the chosen set of parameters that describe the orientations of the bodies in space, the same conclusions apply to more complex systems such as railroad vehicle systems in which the wheelsets rotate about their own axes. Multibody system computer codes are often used to study the stability of these complex systems by linearizing the highly nonlinear equations at different configurations. The eigen solution of the linearized equations is used to draw conclusions about the system stability. Different railroad vehicle system codes, however, employ different sets of orientation coordinates, and as a consequence, the eigen solution results obtained using the linearized equations of the railroad vehicle systems must be carefully analyzed because of their dependence on the set of parameters used to describe the vehicle motion.

8. Summary and conclusions

Computational multibody system algorithms allow for the linearization of the nonlinear system equations of motion at different time points that correspond to different system configurations. The resulting linear equations are used to formulate an eigenvalue problem that can be solved for the eigenvalues and eigenvectors. The eigen solution is often used to shed light on the system stability at different configurations and time points [10–12]. Different multibody system algorithms, however, employ different sets of rotation parameters that define the orientation of the body reference in space. The use of different sets of orientation parameters leads to different forms of the dynamic equations of motion. As a consequence, different multibody system algorithms produce different sets of linear equations for the eigenvalue analysis. Furthermore, different orientation parameters exhibit different dynamic behaviors. For instance, Euler angles can assume any values depending on the load and constraints applied to the system; while the absolute values of Euler parameters cannot exceed one regardless of the forces and constraints applied to the system. As demonstrated in this investigation, the eigenvalue solution depends on the set of coordinates used to describe the dynamics of the multibody system. In the case of Euler angles, for example, rigid body motion can be associated with zero eigenvalues; while this may not be the case when Euler parameters are used. That is, in the case of Euler parameters, rigid body motion is not necessarily associated with zero eigenvalues. A similar comment applies to Rodriguez parameters as demonstrated in Section 6. For this reason, the users of general purpose multibody system computer codes must be careful in interpreting the eigenvalue solution results produced by these codes. The users also must have good knowledge of the set of orientation parameters used by the codes in order to be able to correctly interpret these results.

Acknowledgements

This work was supported by the US Army Research Office, Research Triangle Park, NC; and by the Federal Railroad Administration, Washington, DC. The author would also like to thank Mr. Bassam Hussein of University of Illinois at Chicago for his help in providing some of the formulas used in Section 6 of the paper.

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